ON SOME SPECIAL PROBLEMS OF THERMODYNAMICS OF CONTINUOUS MEDIA

(O NEKOTORYKH SPETSIAL'NYKH VOPROSAKH TERMODINAMIKI SPLOSHNYKH SRED)

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In the construction of mathematical models which have to describe the motion of different types of media (water and other liquids, elastically and plastically deformable materials), the basic relations of these models are frequently formulated without referring to the law of conservation of energy, i.e. the first law of thermodynamics. The system of equations of a model becomes closed, and different problems are formulated and solved. Nevertheless, a model established without the use of thermodynamical concepts cannot be considered to be complete, and the question arises if this model can be justified thermodynamically, i.e. if it does not contradict the basic concepts of thermodynamics.

On the other hand, in order to formulate the system of equations describing the motion of a medium, it is necessary to be able to generalize the thermodynamical relations of a medium which are valid for reversible processes without macroscopic motions to processes with macroscopic motions, usually complicated by dissipative (irreversible) factors. Here, the question arises if the thermodynamics of reversible static processes may be preserved for the problems of motion, or if it should be essentially changed. Obviously, the possibility of preserving the thermodynamics without changes is of considerable interest. It is necessary, therefore, to investigate in each specific case if such a possibility exists.

In this note, a discussion of these problems is given (as it seems that a clear analysis of them does not exist in literature) because they currently have acquired a certain timeliness, especially in the mechanics of deformable solid media.

1. Consider an ideal fluid whose equation of state has the form*

$$p = f(\rho) \tag{1.1}$$

where p is pressure, and ρ is density. From the relation

$$TdS = dE + pdV \tag{1.2}$$

which is a consequence of the first and the second laws of thermodynamics of reversible processes [1] (with T being the temperature, S and E being the entropy and internal energy per unit mass, respectively, and $V = 1/\rho$); the following relation is obtained on the basis of the integrability of dS

$$\left(\frac{\partial E}{\partial V}\right)_T + p = T\left(\frac{\partial p}{\partial T}\right)_V \tag{1.3}$$

Substituting here the equation of state (1.1), we obtain

$$\left(\frac{\partial E}{\partial V}\right)_T = -p = -f\left(\frac{1}{V}\right), \quad \text{or}, \quad E(T, V) = -\int f\left(\frac{1}{V}\right) dV + A(T)$$

where $A(T)$ is an arbitrary function of temperature. But

$$\left(\frac{\partial E}{\partial T}\right)_V = A'(T) = C_V$$

where C_V is the heat capacity at constant volume, which thus proves to be dependent only on temperature. Therefore, we finally obtain the following expression for the internal energy of the medium

$$E(T, V) = \int C_V(T)\, dT - \int f\left(\frac{1}{V}\right) dV \tag{1.4}$$

This medium is called the medium with separable internal energy (energy separates into the part dependent only on temperature and the part dependent only on specific volume).

Substituting (1.4) into (1.2) and integrating, we obtain the expression for entropy

$$S = \int \frac{C_V(T)}{T} dT \tag{1.5}$$

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* The discussion of thermodynamical properties of the medium of this type is given in [2]. Here, this case is considered for the completeness of presentation and in order to exhibit some additional details.

which proves to be a function of temperature only. Similarly, the expressions for all remaining thermodynamical characteristic functions may be obtained. In particular, for free energy Ψ we have

$$\Psi = -\int f\left(\frac{1}{V}\right) dV + \int C_V (T)\, dT - T \int \frac{C_V(T)}{T}\, dT \tag{1.6}$$

i.e. it also separates. Furthermore, because

$$\delta Q = TdS = T\frac{dS}{dT}\, dT = CdT = C_V (T)\, dT$$

all the heat capacities coincide and depend on temperature only.

Finally, the coefficient of thermal expansion α of this medium is equal to zero, because on the basis of (1.1)

$$\alpha \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = 0 \tag{1.7}$$

Conversely, if the coefficient of thermal expansion is equal to zero, then the medium possesses only one coefficient of thermal capacity, separable internal energy, and the equation of state of the type (1.1).

It is interesting to note that, in a medium of the type considered, the isotherms and the adiabatics coincide, and it is impossible to complete a Carnot cycle for this medium.

The equation of state of the type (1.1) is frequently used for the description of motion of water with compressibility being taken into account (see, for example, [3]). In this, the system of equations describing this motion is closed without the use of the equations of energy. We see, that the thermodynamics of reversible processes may be constructed for this medium acoding to the presented scheme. If we assume that viscosity and thermal conductivity are absent, and that the processes occurring in the fluid during the motion are also reversible, i.e. the thermodynamics of equilibrium is applicable, then the model is thermodynamically complete and correct, and the equation of energy assumes the form

$$T\frac{dS}{dt} = C_V(T)\frac{dT}{dt} = 0 \tag{1.8}$$

In this, the mechanical problem is entirely separated from the thermal problem. The system of equations for the determination of the velocity field, the pressure, and the density is closed without Equation (1.8), and they may be solved independently from the latter. After the velocity field is determined, it is possible, if necessary, to solve Equation (1.8) and to find the temperature field.

Another, more frequent case of a medium of the type considered is the model of an incompressible ideal fluid without thermal conductivity. The above statements apply also to this case.

The model of an ideal elastic Hookean solid is an example of the model of a deformable solid medium, which is thermodynamically similar to that considered previously. Considering small static deformations of such a medium as reversible processes, the thermodynamics may be constructed in the following form [4]. The first and the second laws of thermodynamics reduce to the relation

$$TdS = dE - \sigma_{ij}\, d\varepsilon_{ij} \tag{1.9}$$

where S and E are the entropy and internal energy per unit volume, respectively, (because deformations considered in this model are small, these quantities correspond approximately to those taken per unit mass); σ_{ij} are the components of the stress tensor; ϵ_{ij} are the components of the infinitesimal strain tensor. Temperature T and ϵ_{ij} are considered here as the parameters of state.

If the relation between the components of stress and strain tensors is given by Hooke's law

$$\sigma_{ij} = K\varepsilon_{ll}\delta_{ij} + 2G\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{ll}\delta_{ij}\right) \tag{1.10}$$

(repeated indices denote summation), and the elastic constants K and G do not depend on temperature, then from the relation

$$\left(\frac{\partial E}{\partial \varepsilon_{ij}}\right)_T - \sigma_{ij} = -T\left(\frac{\partial \sigma_{ij}}{\partial T}\right)_{\varepsilon_{ij}} \tag{1.11}$$

(which follows from the integrability of dS and is analogous to (1.3)), the relation is obtained

$$\left(\frac{\partial E}{\partial \varepsilon_{ij}}\right)_T = \sigma_{ij}$$

Integrating it and using (1.10), we obtain

$$E(T, \varepsilon_{ij}) = \frac{K}{2}(\varepsilon_{ll})^2 + G\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{ll}\delta_{ij}\right)^2 + A(T) \tag{1.12}$$

If the heat capacity at constant strain will be denoted by $C_{\epsilon_{ij}}$, analogously to C_V, then we have from (1.12)

$$C_{\varepsilon_{ij}} = \left(\frac{\partial E}{\partial T}\right)_{\varepsilon_{ij}} = A'(T) = C_{\varepsilon_{ij}}(T)$$

i.e. this heat capacity depends only on temperature.

The internal energy assumes finally the form

$$E\,(T,\varepsilon_{ij}) = \frac{K}{2}\,(\varepsilon_{ll})^2 + G\,\left(\varepsilon_{ij} - \frac{1}{3}\,\varepsilon_{ll}\delta_{ij}\right)^2 + \int C_{\varepsilon_{ij}}\,(T)\,dT \qquad (1.13)$$

Substituting (1.13) into (1.9) and integrating, we obtain the expression for the entropy

$$S = \int \frac{C_{\varepsilon_{ij}}\,(T)}{T}\,dT \qquad (1.14)$$

which is analogous to the expression (1.5).

Similarly, as in the previous case, we prove that all thermal capacities coincide (the index ϵ_{ij} may be therefore omitted in $C_{\epsilon_{ij}}(T)$), that the coefficient of thermal expansion is equal to zero, and consequently, thermal stresses cannot exist in this medium, etc. It is possible, using (1.13) and (1.14), to write the expressions for all remaining thermodynamical characteristic functions. In particular, we have for free energy, analogously to (1.6),

$$\Psi = \frac{K}{2}\,(\varepsilon_{ll})^2 + G\,(\varepsilon_{ij} - \tfrac{1}{3}\,\varepsilon_{ll}\delta_{ij})^2 + \int C\,(T)\,dT - T\int \frac{C(T)}{T}\,dT \qquad (1.15)$$

In this way, if the elastic constants in the Hooke's law (which plays the role of the equation of state analogously to (1.11) do not depend on temperature, then the complete thermodynamical model of the medium proves to be a model with separable energy and with entropy depending on temperature only.

Assuming, as was done previously, that deformation of the medium in the process of motion is reversible and is described by the same Hooke's law, and that the thermodynamics established for the statical reversible processes is valid also for motion, we obtain a model of the medium which is suitable also for the description of dynamical processes. In this case, the system of equations is also closed without the equation of energy. The equation of energy reduces here to the simple relation

$$T\,\frac{dS}{dt} = C\,\frac{dT}{dt} \approx C\,\frac{\partial T}{\partial t} = 0 \qquad (1.16)$$

It has been shown in the examples considered that the construction of a closed system of equations without the law of conservation of energy does not, in general, contradict this law, and a thermodynamics can be established for these models. But in these examples we had only reversible processes. In the majority of cases, the motion of a medium is accompanied by irreversible, dissipative phenomena.

In establishing the thermodynamics with these phenomena being taken into account, the question arises what is the system of the parameters of state. Is it the same as for the reversible processes, or should new parameters be added, according to the characteristics of dissipative factors? The examples which will be considered in the following show that in some cases the system of the parameters of state and the forms of thermodynamic functions may be retained without changes; in other, more complex,cases thermodynamical functions depend also on additional parameters which, however, are not the thermodynamical parameters of state in the ordinary sense. They characterize the irreversible changes of properties of a medium which occur during irreversible, dissipative processes in this medium.

2. It is generally assumed that, for the model of incompressible viscous heat-conducting fluid, the above thermodynamics with only one parameter of state, temperature T, is valid, that the closed system of mechanical equations (with a temperature-independent coefficient of viscosity) remains preserved without the use of the law of conservation of energy, and that the law of conservation of energy gives an additional equation, i.e. the equation of heat flux (or input). Integrating this equation, the temperature distribution and its changes in time may be determined. From the assumption that the relation

$$TdS = dE = C\,(T)\,dT\left(dV = d\frac{1}{\rho} \equiv 0\right)$$

remains valid, and from the equation of heat flux which has a general form

$$\frac{\delta Q}{\delta t} = \rho\frac{dE}{dt} - \frac{1}{2}\sigma_{ij}e_{ij} \qquad \left(e_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2.1}$$

(where $\delta Q/\delta t$ is the external heat-input rate per unit volume, σ_{ij} are the components of stress tensor, and ϵ_{ij} are the components of strain rate tensor), we have for the case being considered

$$\rho C\,(T)\frac{dT}{dt} = \frac{1}{2}\mu e_{ij}e_{ij} + \operatorname{div}(\varkappa \operatorname{grad} T) \tag{2.2}$$

The above relation contains the assumption that the external heat input is due only to heat conduction (μ and κ are the coefficients of viscosity and thermal conductivity, respectively). After the mechanical problem is solved independently from (2.2), and the velocity field is determined, the temperature distribution may be found from (2.2).

The possibility of this generalization of the model over the cases with viscosity and heat conduction is confirmed by the fact that the results obtained in this way agree with experiments.

We can proceed in the same way also in the case when the fluid is compressible, and the equation of state has the form (1.1). Assuming that the thermodynamics is valid also for motions with viscosity and heat conduction, i.e. that the relations (1.1), (1.4) and (1.5) remain valid, and viscosity is of Newtonian type, with the coefficients of viscosity being independent of temperature, we obtain from Equation (2.1)

$$\rho C(T)\frac{dT}{dt} = \frac{1}{2}\mu e_{ij}e_{ij} - \frac{2}{3}\mu\,(\operatorname{div}\mathbf{V})^2 + \operatorname{div}(\varkappa \operatorname{grad} T) \tag{2.3}$$

The system of dynamical equations remains closed independently of Equation (2.3), which is utilized for the determination of the temperature T, after the fields of density and velocity are found from the solution of dynamical problems. Similarly, as in the previous case, the validity of the constructed model may be checked by experiments. The model itself is correct, dynamically and thermodynamically.

As a third example, let us consider the model of a viscous heat-conducting gas, which is the basis of modern dynamics of gases. It is assumed for the construction of this model that the system of the parameters of state and the form of thermodynamic functions of a moving gas remain the same as for reversible quasi-static processes, i.e. the equations of state remain unchanged, the thermodynamical identity is

$$TdS = dE + pd\frac{1}{\rho}$$

and the equation of heat-flow (2.1) assumes the form

$$\rho\frac{dE}{dt} = \frac{1}{2}\mu e_{ij}e_{ij} - \frac{2}{3}\mu\,(\operatorname{div}\mathbf{V})^2 - p\operatorname{div}\mathbf{V} + \operatorname{div}(\varkappa \operatorname{grad} T) \tag{2.4}$$

Experiment confirms the validity of these assumptions.*

Finally, in a similar manner, the complete model of a linear visco-elastic solid may be constructed, retaining the introduced thermodynamical relations and functions. In so doing, the law of conservation of energy obviously reduces again to the equation of heat flux, which is used for the determination of the temperature field.

In fact, the relations determining an isotropic visco-elastic body have the form [4]

$$\sigma_{ij} = \sigma_{ij}^{e} + \eta\left(e_{ij} - \frac{1}{3}e_{ll}\delta_{ij}\right) + \frac{\zeta}{2}e_{ll}\delta_{ij} \tag{2.5}$$

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* A detailed and very good discussion of these problems may be found in [5] (Chapter II).

where $\sigma_{ij}{}^e$ are "elastic" stresses, related to strain, as previously, by Hooke's law (1.10), and η and ζ are the coefficients of viscosity. Substituting these relations into the equation of heat flux (2.1), and considering (1.10) and (1.13), we obtain

$$\frac{\delta Q}{\delta t} = C(T)\frac{dT}{dt} - \frac{1}{2}\eta e_{ij}e_{ij} + \frac{2}{3}\eta(\operatorname{div}\mathbf{V})^2 - \zeta(\operatorname{div}\mathbf{V})^2 =$$

$$= C(T)\frac{dT}{dt} - \frac{1}{2}\eta\left(e_{ij} - \frac{1}{3}e_{ll}\delta_{ij}\right)^2 - \frac{\zeta}{4}(e_{ll})^2$$

or, assuming $\delta Q/\delta t = \operatorname{div}(\kappa \operatorname{grad} T)$ (isotropic heat conduction)

$$C(T)\frac{dT}{dt} = \frac{1}{2}\eta\left(e_{ij} - \frac{1}{3}e_{ll}\delta_{ij}\right)^2 + \frac{\zeta}{4}(e_{ll})^2 + \operatorname{div}(\varkappa \operatorname{grad} T) \qquad (2.6)$$

If, in addition, the coefficients of viscosity are assumed to be independent of temperature, then the system of dynamical equations is closed, as in previous cases, without the use of the equation of heat flux.

The application of the introduced concepts to the theory of plasticity is of essential and principal significance. The existing theories of plasticity have been developed without the use of the law of conservation of energy and without thermodynamical considerations in general. Therefore, the question of thermodynamical correctness arises also in relation to these theories. It is necessary to note that, on the one hand, the opinions are expressed that the thermodynamical considerations should be the essential basis in the construction of the theory of plasticity, while on the other hand, the complete ignoring of thermodynamics may be observed in the majority of books and papers on plasticity. Both of these points of view are, as we shall see, extreme.

The existing schemes describing the plastic behavior of metals may be completed to full thermodynamical models in the similar way as it was done in the previous examples.

3. Consider, for example, the model of an ideal rigid-plastic material of Saint Venant, Levy, and Mises [6,7]. The basic relations of this model are:

The relation between the deviator of stress tensor σ_{ij}' and the strain rate tensor e_{ij}

$$e_{ij} = \mu\sigma_{ij}' \qquad (3.1)$$

The yield condition, having usually the form

$$\sigma_{ij}'\sigma_{ij}' \equiv 2J_2' = 2k^2 \qquad (3.2)$$

(yield condition of Mises), which makes possible the determination of μ

$$\mu = \frac{\sqrt{I}}{k} \qquad (2I \equiv e_{ij}e_{ij}) \tag{3.3}$$

The condition of incompressibility of the material

$$e_{ii} = 0 \tag{3.4}$$

Substituting these relations into the equation of heat flux (2.1), and carrying out elementary transformations, we obtain

$$\frac{\delta Q}{\delta t} = \frac{dE}{dt} - k\sqrt{I} \tag{3.5}$$

(here and in the following, E and S are referred to unit volume).

The last term in this equation expresses the dissipation of mechanical energy in plastic flow of the material. Entirely analogously to the case of an incompressible fluid, it may be assumed that the only parameter of state in this case is also the temperature T and, consequently,

$$E = E(T) = \int C(T)\, dT \tag{3.6}$$

where $C(T)$ is specific heat capacity of the material,

$$S = \int \frac{C(T)}{T}\, dT \qquad \text{etc.} \tag{3.7}$$

Equation (3.5) reduces to the form

$$C(T)\frac{dT}{dt} = k\sqrt{I} + \operatorname{div}(\varkappa \operatorname{grad} T) \tag{3.8}$$

and may be used for the determination of the temperature field in the flowing plastic medium. If the plastic constant of the material k does not depend on temperature, then the system of mechanical equations proves to be closed without (3.8), and it may be integrated separately.

Consider now the model of the ideal elastic-plastic material of Prandtl and Reuss [6,7]. The basic relations of this model are the following:

The relation between the deviator of the stress tensor σ_{ij}' and the deviator of the strain-rate tensor e_{ij}' is

$$e_{ij}' = \frac{1}{G}\dot{\sigma}_{ij}' + e(\dot{W})\, e(J_2' - k^2)\, \frac{\dot{W}}{k^2}\sigma_{ij}' \tag{3.9}$$

where

$$\dot{W} \equiv \tfrac{1}{2}\sigma_{ij}'e_{ij}', \qquad e(u) = \begin{cases} 1 & (u \geqslant 0) \\ 0 & (u < 0) \end{cases} \qquad \left(\frac{\text{unit step}}{\text{function}}\right) \tag{3.10}$$

and a dot over σ_{ij}' denotes the derivative with respect to time, and (under the condition of small displacements, velocities, and strains) it is simply partial differentiation with respect to time.*

The yield condition of Mises

$$\sigma_{ij}'\sigma_{ij}' \equiv 2J_2' \leqslant 2k^2$$

or, in equivalent form

$$(J_2' - k^2)\, e\,(J_2' - k^2) = 0 \tag{3.11}$$

The condition of the elastic volumetric deformation is

$$\sigma_{ll} = 3K\varepsilon_{ll} \tag{3.12}$$

Using all these relations, we transform the sum $1/2\, \sigma_{ij}\epsilon_{ij}$ in the equation of heat flux (2.1) in the following way:

$$\begin{aligned}
\tfrac{1}{2}\sigma_{ij}e_{ij} &= \tfrac{1}{2}\left(\sigma_{ij}' + \tfrac{1}{3}\sigma_{ll}\delta_{ij}\right)\left(e_{ij}' + \tfrac{1}{3}e_{ll}\delta_{ij}\right) \\
&= \tfrac{1}{2}\sigma_{ij}'e_{ij}' + \tfrac{1}{6}\sigma_{ll}e_{ll} = \tfrac{1}{2}\sigma_{ij}'\left[\frac{1}{G}\dot{\sigma}_{ij}' + e(\dot{W})\,e(J_2' - k^2)\frac{\dot{W}}{k^2}\sigma_{ij}'\right] + \frac{K}{2}\varepsilon_{ll}e_{ll} \\
&= \frac{K}{2}\frac{d(\varepsilon_{ll})^2}{dt} + G\frac{d}{dt}(\varepsilon_{ij}^{e'}\varepsilon_{ij}^{e'}) + e(\dot{W})\,e(J_2' - k^2)\frac{J_2'}{k^2}\dot{W} \\
&= \frac{1}{2K}\frac{d}{dt}\left(\frac{\sigma_{ll}}{3}\right)^2 + \frac{1}{2G}\frac{dJ_2'}{dt} + e(\dot{W})\,e(J_2' - k^2)\frac{J_2'}{k^2}\dot{W}
\end{aligned} \tag{3.13}$$

The components of the deviator of the elastic-strain tensor $\epsilon_{ij}^{e'}$ are here introduced; they are related to the components of the deviator of the stress tensor by the Hooke's law

$$\sigma_{ij}' = 2G\varepsilon_{ij}^{e'} \tag{3.14}$$

In (3.13), the relation $e_{ll} = d\epsilon_{ll}/dt$ is taken into account.

Substituting the expressions (3.13) into Equation (2.1), we have

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* In cases of finite deformation, the determination of time derivatives of the stress tensor in the relations (3.9) of Prandtl and Reuss is not elementary, and it needs special considerations, as has been shown by Prager [8]. See also paper by Sedov [9] on the same subject.

$$\frac{\delta Q}{\delta t} = \rho \frac{dE}{dt} - \frac{K}{2} \frac{d(\varepsilon_{ll})^2}{dt} - G \frac{d}{dt} (\varepsilon_{ij}{}^{e'} \varepsilon_{ij}{}^{e'}) + e(\dot{W})\, e\,(J_2{}' - k^2) \frac{J_2{}'}{k^2} \dot{W}$$
$$= \rho \frac{dE}{dt} - \frac{1}{2K} \frac{d}{dt} \left(\frac{\sigma_{ll}}{3}\right)^2 - \frac{1}{2G} \frac{dJ_2{}'}{dt} + e(\dot{W})\, e\,(J_2{}' - k^2) \frac{J_2{}'}{k^2} \dot{W}. \qquad (3.15)$$

If we assume now that the internal energy E depends only on ϵ_{ll}, $\epsilon_{ij}{}^{e'}$, T (as in the case of an ordinary elastic body), and the elastic constants K and G, and the density ρ do not depend on temperature, and neglecting ϵ_{ll} as compared to 1, we obtain for ρE (internal energy per unit volume) the following expression:

$$E_0 \equiv \rho E = \frac{K}{2} (\varepsilon_{ll})^2 + G \varepsilon_{ij}{}^{e'} \varepsilon_{ij}{}^{e'} + \int C(T)\, dT$$
$$= \frac{1}{2K} \left(\frac{\sigma_{ll}}{3}\right)^2 + \frac{J_2{}'}{2G} + \int C(T)\, dT = \frac{1}{2} \frac{\sigma_{ll}}{3} \varepsilon_{ll} + \frac{1}{2} \sigma_{ij}{}' \varepsilon_{ij}{}^{e'} + \int C(T)\, dT \qquad (3.16)$$

which coincides with (1.13).

For the entropy and the free energy we again obtain the expressions (1.14), (1.15), etc. In this way we again obtain a medium with separable energy and with thermodynamic functions of the same variables as in an ordinary elastic solid, i.e. the consideration of plastic deformation in the ideal elastic-plastic material of Prandtl and Reuss does not contribute new parameters on which thermodynamic functions would depend.

In this sense the ideal plastic body is also "ideal" in the thermodynamic sense; plastic deformation does not influence the thermodynamics of the medium. Plastic deformation causes only the dissipation of mechanical energy, which in the form of heat appears in the equation of heat flux which has finally the form

$$C\,(T) \frac{dT}{dt} = \operatorname{div}\,(\varkappa \operatorname{grad} T) + e\,(\dot{W})\, e\,(J_2{}' - k^2) \frac{J_2{}'}{k^2} \dot{W} \qquad (3.17)$$

This heat flux (the second component in the right-hand side of (3.17)) is obviously always positive, and it exists only if plastic deformation occurs. The system of mechanical equations is here also closed without (3.17), and it is integrated separately while (3.17) determines the field of temperature. Note that for $K \to \infty$ and $G \to \infty$ all the relations of this case transform into the relations of the preceding example (i.e. the relations of a rigid-plastic medium).

Let us finally consider a more complex model of a plastic material, i.e. a model with strain hardening.

In the process of plastic deformation, generally speaking, changes of elastic limits occur, and also elastic constants change. This last phenomenon is accompanied by the appearance of anisotropy in the elastic properties of material. Therefore, all the changes of the elastic properties of material, caused by plastic deformation, may be considered as strain hardening in a broader sense.

The example of an ideally plastic medium considered above indicates that the thermodynamic functions for such a medium depend, in addition to the temperature, on the invariants of the elastic-stress tensor and the constant parameters which characterize the elastic properties and the yield point of the material. Thus, because these constant parameters do not depend on the characteristics of plastic deformation, the entire thermodynamics of the medium does not depend on the latter. If these parameters change in the process of plastic deformation, then also the form of thermodynamic functions changes, i.e. these functions will depend on the characteristics of plastic deformations. Using a simple model of a plastic material with strain hardening, we shall explain what thermodynamical consequences result from the consideration of these dependences.

Consider the model of the isotropic strain-hardening material of Prandtl and Reuss with the yield condition of Mises [7]. The basic relations of this model may be written in the form

$$e_{ij}' = \frac{\dot{\sigma}_{ij}'}{G} + \frac{3\dot{\xi}}{\sigma^\circ}\sigma_{ij}', \qquad \sigma^\circ \equiv \sqrt{3J_2'}, \qquad \sigma_{ll} = 3K\varepsilon_{ll} \tag{3.18}$$

$$\dot{\xi} \equiv \frac{d\xi}{dt} = \frac{\dot{\sigma}^\circ}{H'(\xi)}\, e[\sigma^\circ - H(\xi)]\, e(\dot{\sigma}^\circ), \qquad [\sigma^\circ - H(\xi)]\, e[\sigma^\circ - H(\xi)] = 0$$

where e is the previously introduced unit step function; $H(\xi)$ is the strain-hardening function; ξ is the strain-hardening parameter, i.e. the measure of plastic deformation, introduced by Odqvist. It is necessary to notice that in order to determine the quantity ξ, which on the basis of (3.18) may be expressed in terms of the plastic-strain-rate tensor, its infinitesimal increments in the process of active plastic deformation have to be integrated; however, the introduction of ξ as a new parameter with a differential equation for ξ (see (3.18)) reduces all the relations of the model to purely differential relations, excluding thus the dependence of any element of the model on the "history of deformation". The possibility and the necessity of introducing additional parameters with corresponding differential equations to achieve the exclusion of so-called "history of deformation" from the model has been shown by

Sedov. In this case, ξ is such a (only one) parameter.*

Substituting now the relations (3.18) into the equation of heat flux (2.1) we have to consider that, in addition to the temperature and the invariants of the tensor of elastic deformation, the internal energy may depend also on the parameter ξ. It is necessary, however, to keep in sight the fact that because the parameter ξ changes only if plastic deformation occurs, i.e. only in irreversible processes, the thermodynamical equalities which are valid only for reversible processes should be written with $d\xi = 0$. In particular, for reversible processes we have the thermodynamical equality in the form (1.9), where the differentials of S and E are determined for $\xi =$ const. Considering Hooke's law and assuming that elastic constants do not depend on temperature, this results in the expression for internal energy

$$E = \frac{1}{2K}\left(\frac{\sigma_{ll}}{3}\right)^2 + \frac{J_2'}{2G} + \int C(T, \xi)\, dT + E_0(\xi) \tag{3.19}$$

and for entropy

$$S = \int \frac{C(T, \xi)}{T}\, dT + S_0(\xi) \tag{3.20}$$

(where E_0 and S_0 are arbitrary functions of ξ). The expressions (3.19) and (3.20) differ from (1.13) (or (3.16)) and (1.14) only by possible dependence on ξ. Entropy, as well as heat capacity, prove to be functions of T and ξ only. Internal energy depends on ξ not only through $E_0(\xi)$ and $C(T, \xi)$, but also through the dependence of the elastic constants, $K(\xi)$ and $G(\xi)$, on ξ, which is, in this way, admissible for the model.

Assuming again that the relations (3.19) and (3.20) remain valid also for the cases of progressing plastic deformation and existing thermal conduction, i.e. for the cases of irreversible processes, we substitute the relations (3.18) and (3.19) into (2.1) and, after some transformations, we obtain

$$C(T, \xi)\frac{dT}{dt} = \operatorname{div}(\varkappa \operatorname{grad} T) + \left(\sigma^{\circ} - \frac{\partial E}{\partial \xi}\right)\frac{d\xi}{dt} \tag{3.21}$$

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* Several parameters may exist in the case of a more complex strain-hardening plastic medium where the process of plastic deformation is accompanied by anisotropy of its elastic properties and the change of elastic limits (which is, in general, also anisotropic). The construction of thermodynamics for this case may be accomplished essentially in the same way as in the simpler case considered here.

The first component in Equation (3.21) determines the heat flux due to thermal conductivity, and the second component determines the heat flux due to the dissipation of mechanical energy.

It is interesting to note that not all the work of plastic deformation $\sigma^0 d\xi/dt$ is converted into heat, but only a part of it

$$\sigma^{\circ}\frac{d\xi}{dt}-\frac{\partial E}{\partial \xi}\frac{d\xi}{dt}$$

i.e. only the excess over the work which increases the internal energy E in connection with changes of ξ.

This increase of internal energy may be treated as the energy taken by the material for irreversible transformations (irreversible changes of structure) which occur during plastic deformation. Because $d\xi/dt$ is positive (see (3.18)), the work of plastic deformation $\sigma^0 d\xi/dt$ is positive.

It may be shown that, in addition, the excess of this work over the energy of transformation should also be positive.

Integrating the expression for dS/dt over a volume Ω, and using (3.20) and (3.21), we obtain

$$\frac{dS_\Omega}{dt} \equiv \frac{d}{dt}\int_\Omega S d\tau = \int_\Sigma \frac{q_n}{T} d\Sigma + \int_\Omega \frac{\varkappa}{T^2}(\operatorname{grad} T)^2 d\tau + \int_\Omega \frac{1}{T}\left(\sigma^{\circ} - \frac{\partial E}{\partial}\right)\frac{d\xi}{dt} d\tau + \int_\Omega \frac{\partial S}{\partial \xi}\frac{d\xi}{dt} d\tau \tag{3.22}$$

where q_n is the normal component of the heat-flux vector. Considering that the surface Σ, bounding the volume Ω, is thermally insulated (this may be principally realized including into the medium, at the surface Σ, a thin layer of an ideal insulating material) we obtain on the basis of the second law of thermodynamics

$$\frac{dS_\Omega}{dt} = \int_\Omega \frac{\varkappa}{T^2}(\operatorname{grad} T)^2 d\tau + \int_\Omega \frac{1}{T}\left(\sigma^{\circ} - \frac{\partial E}{\partial \xi}\right)\frac{d\xi}{dt} d\tau + \int_\Omega \frac{\partial S}{\partial \xi}\frac{d\xi}{dt} d\tau > 0 \qquad (3.23)$$

Each of the written integrals should be positive, because they represent increments of the entropy in a finite volume caused by three independent irreversible phenomena: heat conduction, heat evolution by dissipation of plastic work (second integral), and entropy increment caused by the irreversible increasing of the parameter ξ (third integral). The quantity $(\partial S/\partial \xi)/(d\xi/dt)$ determines the entropy increment caused by irreversible changes of the material which result in the changes of ξ. Since this entropy increment is positive and $d\xi/dt > 0$, then also

$\partial S/\partial \xi > 0$. Thus, the third integral is positive independently of the others (this independence may be formally noted in the independence of functions $E_0(\xi)$ and $S_0(\xi)$). The second integral should be also positive, it determines the entropy increment caused by internal heat production in irreversible plastic deformation.

Because the volume Ω is arbitrary, the following inequality should be satisfied in the region of deformation.

$$\sigma^\circ - \frac{\partial E}{\partial \xi} = H - \frac{\partial E}{\partial \xi} \geqslant 0 \qquad (3.24)$$

which means that the excess of plastic work over the energy of conversion is positive.

The functions $E_0(\xi)$ and $S_0(\xi)$ may be determined by establishing the relation between the changes of ξ and the microscopic changes of structure for which the complementary energy of transformation $E_0(\xi)$ is used and which are accompanied by the existence of the entropy $S_0(\xi)$.

The last example considered shows that the dependence of thermodynamic functions on the parameters characterizing strain hardening, i.e. the influence of plastic deformation on elastic properties of the material, introduces certain complexities into the thermodynamics of the medium, but it does not cause any principal difficulties.

The aim of the discussion presented here was to show that the use of thermodynamics for the construction of different models describing the motion of fluids and, particularly, plastically deformable solid bodies results in certain relations and concepts which make these models thermodynamically complete. Unfortunately, these concepts are of limited usefulness in the construction of the mechanical parts of models, and they do not essentially simplify this main problem. Nevertheless, it is necessary to have always a clear picture of the thermodynamical meaning of the model constructed and to be able to complement this model thermodynamically.

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